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INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH
TECHNOLOGYON BINARY QUADRATIC EQUATION $y^2 = 35x^2 + 29$ S.Mallika^{*1} & V.Surya²^{*} Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-2,
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ABSTRACT

The binary quadratic equation represented by the positive pellian $y^2 = 35x^2 + 29$ is analyzed for its distinct integer solutions. A few interesting relation among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas

KEYWORDS: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

1. INTRODUCTION

The binary quadratic Diophantine equations are rich in variety. The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non square positive integer has been satisfied by various mathematician for its non-trivial integral solution. When D takes different integral values [1-4]. In [5-15] the binary quadratic non-homogeneous equation representing hyperbolas respectively are studied for their non-zero integral solutions. This communication concerns with yet another binary quadratic equation given by $y^2 = 35x^2 + 29$. The recurrence relation satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is

$$y^2 = 35x^2 + 29 \quad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 8$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 35x^2 + 1$$

whose smallest positive integer solution is

$$\tilde{x}_0 = 1, \tilde{y}_0 = 6$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{35}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (6 + \sqrt{35})^{n+1} + (6 - \sqrt{35})^{n+1}$$



$$g_n = (6 + \sqrt{35})^{n+1} - (6 - \sqrt{35})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{4}{\sqrt{35}} g_n$$

$$y_{n+1} = 4 f_n + \frac{\sqrt{35}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 12x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 12y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	8
0	14	83
1	167	988
2	1990	11773
3	23713	140288
4	282566	1671683

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} & y_{n+1} values are alternatively odd and even .

2. Relations among the solutions

- $x_{n+3} - 12x_{n+2} + x_{n+1} = 0$
- $y_{n+1} - x_{n+2} + 6x_{n+1} = 0$
- $y_{n+2} - 6x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 71x_{n+2} + 6x_{n+1} = 0$
- $12y_{n+1} + 71x_{n+1} - x_{n+3} = 0$
- $2y_{n+2} + x_{n+1} - x_{n+3} = 0$
- $12y_{n+3} + x_{n+1} - 71x_{n+3} = 0$
- $y_{n+2} - 6y_{n+1} - 35x_{n+1} = 0$
- $y_{n+3} - 71y_{n+1} - 420x_{n+1} = 0$
- $6y_{n+1} - y_{n+2} + 35x_{n+1} = 0$
- $6y_{n+3} - 71y_{n+2} - 35x_{n+1} = 0$
- $y_{n+1} + 71x_{n+2} - 6x_{n+3} = 0$
- $y_{n+2} + 6x_{n+1} - x_{n+3} = 0$
- $y_{n+3} + x_{n+2} - 6x_{n+3} = 0$



- $6y_{n+2} - y_{n+1} - 35x_{n+2} = 0$
- $6x_{n+2} - x_{n+3} + y_{n+2} = 0$
- $12y_{n+2} - y_{n+1} - y_{n+3} = 0$
- $35x_{n+1} + 71y_{n+2} - 6y_{n+3} = 0$
- $35x_{n+3} + y_{n+2} - 6y_{n+3} = 0$
- $y_{n+1} + 420x_{n+3} - 71y_{n+3} = 0$

3. Each of the following expressions represents a nasty number

- $\frac{1}{29}(96x_{2n+3} - 996x_{2n+2} + 348)$
- $\frac{1}{87}(24x_{2n+4} - 2964x_{2n+2} + 1044)$
- $\frac{1}{29}(96y_{2n+2} - 420x_{2n+2} + 348)$
- $\frac{1}{87}(48y_{2n+3} - 2940x_{2n+2} + 1044)$
- $\frac{1}{2059}(96y_{2n+4} - 70140x_{2n+2} + 24708)$
- $\frac{1}{29}(996x_{2n+4} - 11856x_{2n+3} + 348)$
- $\frac{1}{87}(498y_{2n+2} - 210x_{2n+3} + 1044)$
- $\frac{1}{29}(996y_{2n+3} - 5880x_{2n+3} + 348)$
- $\frac{1}{87}(498y_{2n+4} - 35070x_{2n+3} + 1044)$
- $\frac{1}{2059}(11856y_{2n+2} - 420x_{2n+4} + 24708)$
- $\frac{1}{87}(5928y_{2n+3} - 2940x_{2n+4} + 1044)$
- $\frac{1}{29}(11856y_{2n+4} - 70140x_{2n+4} + 348)$
- $\frac{1}{29}(168y_{2n+2} - 12y_{2n+3} + 348)$
- $\frac{1}{87}(501y_{2n+2} - 3y_{2n+4} + 1044)$
- $\frac{1}{29}(2004y_{2n+3} - 168y_{2n+4} + 348)$



**4. Each of the following expressions represents a cubical integer**

- $\frac{1}{29}[16x_{3n+4} - 166x_{3n+3} + 48x_{n+2} - 498x_{n+1}]$
- $\frac{1}{87}[4x_{3n+5} - 494x_{3n+3} + 12x_{n+3} - 4482x_{n+1}]$
- $\frac{1}{29}[16y_{3n+3} - 70x_{3n+3} + 48y_{n+1} - 210x_{n+1}]$
- $\frac{1}{87}[8y_{3n+4} - 490x_{3n+3} + 24y_{n+2} - 1470x_{n+1}]$
- $\frac{1}{2059}[16y_{3n+5} - 11690x_{3n+3} + 48y_{n+3} - 35070x_{n+1}]$
- $\frac{1}{29}[166x_{3n+5} - 1976x_{3n+4} + 498x_{n+3} - 5928x_{n+2}]$
- $\frac{1}{87}[83y_{3n+3} - 35x_{3n+4} + 249y_{n+1} - 105x_{n+2}]$
- $\frac{1}{29}[166y_{3n+4} - 980x_{3n+4} + 498y_{n+2} - 2940x_{n+2}]$
- $\frac{1}{87}[83y_{3n+5} - 5845x_{3n+4} + 249y_{n+3} - 17535x_{n+2}]$
- $\frac{1}{2059}[1976y_{3n+3} - 70x_{3n+5} + 5928y_{n+1} - 210x_{n+3}]$
- $\frac{1}{87}[988y_{3n+4} - 490x_{3n+5} + 2964y_{n+2} - 1470x_{n+3}]$
- $\frac{1}{29}[1976y_{3n+5} - 11690x_{3n+5} + 5928y_{n+3} - 35070x_{n+3}]$
- $\frac{1}{29}[28y_{3n+3} - 2y_{3n+4} + 84y_{n+1} - 6y_{n+2}]$
- $\frac{1}{174}[167y_{3n+3} - y_{3n+5} + 501y_{n+1} - 3y_{n+3}]$
- $\frac{1}{29}[334y_{3n+4} - 28y_{3n+5} + 1002y_{n+2} - 84y_{n+3}]$

5. Each of the following expressions represents a bi-quadratic integer

- $\frac{1}{29}[16x_{4n+5} - 166x_{4n+4} + 64x_{2n+3} - 664x_{2n+2} + 174]$
- $\frac{1}{87}[4x_{4n+6} - 494x_{4n+4} + 16x_{2n+4} - 1976x_{2n+2} + 522]$
- $\frac{1}{29}[16y_{4n+4} - 70x_{4n+4} + 64y_{2n+2} - 280x_{2n+2} + 174]$
- $\frac{1}{87}[8y_{4n+5} - 490x_{4n+4} + 32y_{2n+3} - 1960x_{2n+2} + 522]$





- $\frac{1}{2059}[16y_{4n+6} - 11690x_{4n+4} + 64y_{2n+4} - 46760x_{2n+2} + 12354]$
- $\frac{1}{29}[166x_{4n+6} - 1976x_{4n+5} + 664x_{2n+4} - 7904x_{2n+3} + 174]$
- $\frac{1}{87}[83y_{4n+4} - 35x_{4n+5} + 140x_{2n+3} - 332y_{2n+2} + 522]$
- $\frac{1}{29}[166y_{4n+5} - 980x_{4n+5} + 664y_{2n+3} - 3920x_{2n+3} + 174]$
- $\frac{1}{87}[83y_{4n+6} - 5845x_{4n+5} + 332y_{2n+4} - 23380x_{2n+3} + 522]$
- $\frac{1}{2059}[1976y_{4n+4} - 70x_{4n+6} + 7904y_{2n+2} - 280x_{2n+4} + 12354]$
- $\frac{1}{87}[988y_{4n+5} - 490x_{4n+6} + 3952y_{2n+3} - 1960x_{2n+4} + 522]$
- $\frac{1}{29}[1976y_{4n+6} - 11690x_{4n+6} + 7904y_{2n+4} - 46760x_{2n+4} + 174]$
- $\frac{1}{29}[28y_{4n+4} - 2y_{4n+5} + 112y_{2n+2} - 8y_{2n+3} + 174]$
- $\frac{1}{174}[167y_{4n+4} - y_{4n+6} + 668y_{2n+2} - 4y_{2n+4} + 1044]$
- $\frac{1}{29}[334y_{4n+5} - 28y_{4n+6} + 1336y_{2n+3} - 112y_{2n+4} + 174]$

6. Each of the following expressions represents a quintic integer

- $\frac{1}{29}[16x_{5n+6} - 166x_{5n+5} + 80x_{3n+4} - 830x_{3n+3} + 160x_{n+2} - 1660x_{n+1}]$
- $\frac{1}{87}[4x_{5n+7} - 494x_{5n+5} + 20x_{3n+5} - 2470x_{3n+3} + 40x_{n+3} - 4940x_{n+1}]$
- $\frac{1}{29}[16y_{5n+5} - 70x_{5n+5} + 80y_{3n+3} - 350x_{3n+3} + 160y_{n+1} - 700x_{n+1}]$
- $\frac{1}{87}[8y_{5n+6} - 490x_{5n+5} + 40y_{3n+4} - 2450x_{3n+3} + 80y_{n+2} - 4900x_{n+1}]$
- $\frac{1}{2059}[16y_{5n+7} - 11690x_{5n+5} + 80y_{3n+5} - 58450x_{3n+3} + 160y_{n+3} - 116900x_{n+1}]$
- $\frac{1}{29}[166x_{5n+7} - 1976x_{5n+6} + 830x_{3n+5} - 9880x_{3n+4} + 1660x_{n+3} - 19760x_{n+2}]$
- $\frac{1}{87}[83y_{5n+5} - 35x_{5n+6} + 415y_{3n+3} - 175x_{3n+4} + 830y_{n+1} - 350x_{n+2}]$
- $\frac{1}{29}[166y_{5n+6} - 980x_{5n+6} + 830y_{3n+4} - 4900x_{3n+4} + 1660y_{n+2} - 9800x_{n+2}]$



- $\frac{1}{87} [83y_{5n+7} - 5845x_{5n+6} + 415y_{3n+5} - 29225x_{3n+4} + 830y_{n+3} - 58450x_{n+2}]$
- $\frac{1}{2059} [1976y_{5n+5} - 70x_{5n+7} + 9880y_{3n+3} - 350x_{3n+5} + 19760y_{n+1} - 700x_{n+3}]$
- $\frac{1}{87} [988y_{5n+6} - 490x_{5n+7} + 4940y_{3n+4} - 2450x_{3n+5} + 9880y_{n+2} - 4900x_{n+3}]$
- $\frac{1}{29} [1976y_{5n+7} - 11690x_{5n+7} + 9880y_{3n+5} - 58450x_{3n+5} + 19760y_{n+3} - 116900x_{n+3}]$
- $\frac{1}{29} [28y_{5n+5} - 2y_{5n+6} - 10y_{3n+4} + 140y_{3n+3} + 280y_{n+1} - 20y_{n+2}]$
- $\frac{1}{174} [167y_{5n+5} - y_{5n+7} - 5y_{3n+5} + 835y_{3n+3} - 10y_{n+3} + 1670y_{n+1}]$
- $\frac{1}{29} [334y_{5n+6} - 28y_{5n+7} + 1670y_{3n+5} - 140y_{3n+4} + 3340y_{n+2} - 28y_{n+3}]$

Remarkable observations:

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbola	(X, Y)
1	$X^2 - 35Y^2 = 841$	$(8x_{n+2} - 83x_{n+1}, 14x_{n+1} - x_{n+2})$
2	$X^2 - 35Y^2 = 121104$	$(8x_{n+3} - 988x_{n+1}, 167x_{n+1} - x_{n+3})$
3	$X^2 - 35Y^2 = 841$	$(8y_{n+1} - 35x_{n+1}, 8x_{n+1} - y_{n+1})$
4	$X^2 - 35Y^2 = 30276$	$(8y_{n+2} - 490x_{n+1}, 83x_{n+1} - y_{n+2})$
5	$X^2 - 35Y^2 = 4239481$	$(8y_{n+3} - 5845x_{n+1}, 988x_{n+1} - y_{n+3})$
6	$X^2 - 35Y^2 = 841$	$(83x_{n+3} - 988x_{n+2}, 167x_{n+2} - 14x_{n+3})$
7	$X^2 - 35Y^2 = 30276$	$(83y_{n+1} - 35x_{n+2}, 8x_{n+2} - 14y_{n+1})$
8	$X^2 - 35Y^2 = 841$	$(83y_{n+2} - 490x_{n+2}, 83x_{n+2} - 14y_{n+2})$
9	$X^2 - 35Y^2 = 30276$	$(83y_{n+3} - 5845x_{n+2}, 988x_{n+2} - 14y_{n+3})$
10	$X^2 - 35Y^2 = 4239481$	$(988y_{n+1} - 35x_{n+3}, 8x_{n+3} - 167y_{n+1})$
11	$X^2 - 35Y^2 = 30276$	$(988y_{n+2} - 490x_{n+3}, 83x_{n+3} - 167y_{n+3})$
12	$X^2 - 35Y^2 = 841$	$(988y_{n+3} - 5845x_{n+3}, 988x_{n+3} - 167y_{n+3})$
13	$35X^2 - Y^2 = 29435$	$(14y_{n+1} - y_{n+2}, 8y_{n+2} - 83y_{n+1})$
14	$35X^2 - Y^2 = 4238640$	$(167y_{n+1} - y_{n+3}, 8y_{n+3} - 988y_{n+1})$
15	$35X^2 - Y^2 = 29435$	$(167y_{n+2} - 14y_{n+3}, 83y_{n+3} - 988y_{n+2})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:



Table: 3 Parabolas

S. No	Parabola	(X, Y)
1	$29X - 70Y^2 = 841$	$(8x_{2n+3} - 83x_{2n+2}, 14x_{n+1} - x_{n+2})$
2	$174X - 35Y^2 = 60552$	$(8x_{2n+4} - 988x_{2n+2}, 167x_{n+1} - x_{n+3})$
3	$29X - 70Y^2 = 841$	$(8y_{2n+2} - 35x_{2n+2}, 8x_{n+1} - y_{n+1})$
4	$87X - 35Y^2 = 15138$	$(8y_{2n+3} - 490x_{2n+2}, 83x_{n+1} - y_{n+2})$
5	$2059X - 35Y^2 = 4239481$	$(8y_{2n+4} - 5845x_{2n+2}, 988x_{n+1} - y_{n+3})$
6	$29X - 70Y^2 = 841$	$(83x_{2n+4} - 988x_{2n+3}, 167x_{n+2} - 14x_{n+3})$
7	$87X - 35Y^2 = 15138$	$(83y_{2n+2} - 35x_{2n+3}, 8x_{n+2} - 14y_{n+1})$
8	$29X - 70Y^2 = 841$	$(83y_{2n+3} - 490x_{2n+3}, 83x_{n+2} - 14y_{n+2})$
9	$87X - 35Y^2 = 15138$	$(83y_{2n+4} - 5845x_{2n+3}, 988x_{n+2} - 14y_{n+3})$
10	$2059X - 70Y^2 = 4239481$	$(988y_{2n+2} - 35x_{2n+4}, 8x_{n+3} - 167y_{n+1})$
11	$87X - 35Y^2 = 15138$	$(988y_{2n+3} - 490x_{2n+4}, 83x_{n+3} - 167y_{n+2})$
12	$29X - 70Y^2 = 841$	$(988y_{2n+4} - 5845x_{2n+4}, 988x_{n+3} - 167y_{n+3})$
13	$1015X - 2Y^2 = 29435$	$(14y_{2n+2} - y_{2n+3}, 83y_{n+1} - 8y_{n+2})$
14	$6090X - Y^2 = 2119320$	$(167y_{2n+2} - y_{2n+4}, 988y_{n+1} - 8y_{n+3})$
15	$1015X - 2Y^2 = 29435$	$(167y_{2n+3} - 14y_{2n+4}, 988y_{n+2} - 83y_{n+3})$

3. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola, represented by positive pell equation is given by $y^2 = 35x^2 + 29$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Positive Pell equations and determine their integer solutions along with suitable properties.

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